New counterexamples to the birational Torelli thm for CY manifolds

1) Torelli-type theorems

- **Thm** (Torelli '13) C, C' smooth projective curve

(Abel-Jacobi maps)

$$\begin{array}{ccccc} p \in C & \xrightarrow{f_p} & J(C) & = & H^0(C, \Omega^1_C)^\vee / H_1(C, \mathbb{Z}) \\ & & \big\downarrow \beta \text{ iso} & & \\ p' \in C' & \xrightarrow{f_{p'}} & J(C') & = & H^0(C', \Omega^1_{C'})^\vee / H_1(C', \mathbb{Z}) \end{array}$$

$\leadsto$ $\exists$ isomorphism $\alpha : C \to C'$ s.t. the diagram commutes (up to sign & translation of β)

"cohomology data determines the isomorphism class of curves"

- K3 surfaces

Thm (Global Torelli theorem [Piatetskii-Shapiro - Shafarevich '71, Burns - Rapoport '75])

Let X, Y be K3 (projective). Then $X \simeq Y$ iff $\exists\ \varphi : H^2(X, \mathbb{Z}) \xrightarrow{\simeq} H^2(Y, \mathbb{Z})$ iso of Hodge structures preserving the intersection pairing. (Hodge isometry)

Higher dimension: two directions

- Hyperkähler varieties: COUNTEREXAMPLE [Namikawa, '00]
  $\rightsquigarrow$ a weaker statement

<u>Thm</u> [Verbitsky '13] ("birational Torelli for HK")

X, Y HK, $H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(Y,\mathbb{Z})$ preserving

BBF + other assumptions $\Longrightarrow X \overset{\text{bir}}{\longleftarrow\!\!-\!-\!\!\rightarrow} Y$

- CY varieties: COUNTEREXAMPLES
  - [Szendrői '00] construction in weighted projective spaces
  - [Aspinwall-Morrison '94], [Szendrői '04] resolutions of quotients of special quintic 3folds by finite groups

<u>Guess</u> ("birational" Torelli thm)

X, Y CYn with $\varphi : H^n(X,\mathbb{Z}) \xrightarrow{\sim} H^n(Y,\mathbb{Z})$
Hodge isometry of middle cohomology
(so-called "Hodge-equivalence"). Then $X \dashleftarrow\dashrightarrow Y$

False!

. Counterexamples in dim = 3
[Ottem-Rennemo '18], [Borisov-Căldăraru-Perry '18]
. in dim = 5 [Manivel '19]
. in dim $4n^2-1$, $n \in \mathbb{N}$ [R. '22]

2) The counterexample for CY3's (and CY5's)

$G(2,V) \hookrightarrow \mathbb{P}^9$ Grassmannian

$g \in PGL(\Lambda^2 V)$ aut. of $\mathbb{P}^9$ (general)

$\rightsquigarrow$ $X := G(2,V) \cap g\,G(2,V) \subset \mathbb{P}^9$ "intersection of general translates"

$Y := G(2,V)^\vee \cap (g\,G(2,V))^\vee$

under the choice of an isomorphism $V_5 \simeq V_5^\vee$ sending $\mathbb{P}(\Lambda^2 V_5)$ to $\mathbb{P}(\Lambda^2 V_5^\vee)$

$= G(2,V) \cap g^{-T} G(2,V)$

<u>Thm</u> ([OR], [BCP]) derived eq., non bir., Hodge equivalent.

(For CY5's: $X := OG(5,10)^+ \cap g\, OG(5,10)^+ \subset \mathbb{P}^{15}$
$Y := OG(5,10)^+ \cap g^{-T} OG(5,10)^+ \subset \mathbb{P}^{15\vee}$)

- Back to the CY3's: degeneration of the family

$\varphi_g : V^\vee \otimes U \to V \otimes \mathcal{O}(1)$ skew-symm. map of v.b.'s on $G(2,5)$ $\left(\varphi_g \in H^0(G(2,5), \Lambda^2 V \otimes \mathcal{O}(1))\right)$

$X_g = D_2(\varphi_g) \subset G(2,5)$ degeneracy locus

$\varphi_g \in H^0\left(G(2,V), \Lambda^2 V \otimes \mathcal{O}(1)\right) \simeq \Lambda^2 V \otimes \Lambda^2 V^\vee \simeq \mathrm{Hom}(\Lambda^2 V, \Lambda^2 V)$

$\rightsquigarrow$ $g \in GL(\Lambda^2 V)$ as an element of $\mathrm{Hom}(\Lambda^2 V, \Lambda^2 V)$

and φ_g the corresp. section.

[Kapustka '13] divisor in the family
[Inoue - Ito - Miura '16]

$$X_0 = Z(s) \text{ w/ } s \in H^0(G(2,V), Q^\vee(2))$$

A simpler duality picture:

$$M = Z\big(\sigma \in H^0(\underbrace{\mathcal{O}(1) \boxtimes \mathcal{O}(1)}_{\mathcal{O}(1,1)})\big)$$

$$\begin{tikzcd}
 & M \arrow[d, phantom, "\cap" rotate=-90] & \\
\mathbb{P}(Q^\vee(2)) = & F(2,3,5) \arrow[dl, "p"'] \arrow[dr, "q"] & = \mathbb{P}(U(2)) \\
X \subset G(2,5) & & G(3,5) \supset Y
\end{tikzcd}$$

$$X = Z(s \in H^0(Q^\vee(2))) \qquad Y = Z(s' \in H^0(U(2)))$$

Thm [Kapustka, R. '17] for σ general, X and Y are:

- derived eq
- Hodge eq (by [OR] & [BCP])
- non birational

The main construction

$$
\begin{tikzcd}
 & M \arrow[d, phantom, "\cap"] & \\
 & F := F(n,n+1,2n+1) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \subset G(n,2n+1) & & G(n+1,2n+1) \supset Y
\end{tikzcd}
$$

$\sigma \in H^0(F, \mathcal{O}(1,1))$ general

$M := Z(\sigma)$

$X := Z(p_*\sigma)$, $Y := Z(q_*\sigma)$] sm.

X, Y CY of dim n^2-1

<u>Obs</u> this is NOT a degeneration of some $G(n,2n+1) \cap g G(n,2n+1)$ - family!

- $h^{1,n^2-2}(X) - \left(h^0(G(n,2n+1), Q^\vee(2)) - \dim \mathrm{Aut}(G(n,2n+1))\right) = \begin{cases} 1 & n=2 \\ 0 & n>2 \end{cases}$
- $Q^\vee(2) \not\cong N_{G(n,V)|\mathbb{P}(\Lambda^n V)}$ for $n>2$
- $G(n,V) \cap g G(n,V) = \emptyset$ for general g

3) Hodge equivalence

$p : \mathbb{P}(Q^\vee(2)) \longrightarrow G(k, 2k+1)$

$q : \mathbb{P}(U(2)) \longrightarrow G(k+1, 2k+1)$

$$\begin{tikzcd} & M \arrow[d, phantom, "\cap"] & \\ & F := F(k,k+1,2k+1) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \subset G(k,2k+1) & & G(k+1,2k+1) \supset Y \end{tikzcd}$$

$\bar{p} := p|_M \quad \bar{q} := q|_M .$

$$\bar{p}^{-1}(x) \simeq \begin{cases} \mathbb{P}^{k-1} & x \in G(k,2k+1) \setminus X \\ \mathbb{P}^{k} & x \in X \end{cases}$$ same for $\bar{q}$

$$\rightsquigarrow H^{k^2+2k-1}(M, \mathbb{Z}) \simeq H^{k^2-1}(X, \mathbb{Z}) \oplus \bigoplus_{m=1}^{k} H^{k^2-1+2m}(G(k,V), \mathbb{Z})$$

$$\simeq H^{k^2-1}(Y, \mathbb{Z}) \oplus \bigoplus_{m=1}^{k} H^{k^2-1+2m}(G(k+1,V), \mathbb{Z})$$

gen. of [Voisin] (see [Bernardara - Fatighenti - Manivel '19])

Now assume n^2-1 is odd. Then the terms from the Grassmannians disappear!

$$H^{n^2-1}(X,\mathbb{Z}) \simeq H^{n^2-1}(Y,\mathbb{Z})$$

Is it an isometry?

$$H^{n^2-1}(X,\mathbb{Z}) \hookrightarrow H^{n^2+2n-1}(M,\mathbb{Z})$$

$$a \longmapsto j_{x*}p_E^* a$$

$$\begin{tikzcd}
E_x \arrow[r, hook, "j_x"] \arrow[d, "p_E"'] & M \arrow[r, hook] \arrow[d, "\bar{p}"'] & F \arrow[dl, "p"] \\
X \arrow[r, hook] & G(n,V) &
\end{tikzcd}$$

$$(j_{x*}p_E^*a \cdot j_{x*}p_E^*b)_M = j_{x*}(\underbrace{j_x^* j_{x*} p_E^* a}_{\text{focus on this}} \cdot p_E^* b)_{E_x}$$

"self-int. formula"

coh. of $G(k,v)$

$$j_X^{*}\, j_{X*}\, p_E^{*} a = p_E^{*} a \cdot c_k(\mathcal{N}_{E_X|M}) = \underbrace{\alpha\, \xi^{k}}_{\text{only surviving term}} + \sum_{i=1}^{k} \beta_i\, \xi^{k-i}\, S_i$$

$$\alpha = \deg\left(\mathcal{N}_{E_X|M}\big|_f\right) \qquad f = \bar{p}^{-1}(x) \text{ w/ } x \in X, \quad f \simeq \mathbb{P}^k$$

$$0 \to \mathcal{N}_{E_X|M}\big|_f \to \mathcal{N}_{E_X|F}\big|_f \to \mathcal{O}(1,1)\big|_f \to 0$$

$$\mathcal{N}_{E_X|F}\big|_f = \mathcal{O}^{\oplus(k+1)}, \qquad \mathcal{O}(1,1)\big|_f = \mathcal{O}_{\mathbb{P}^k}(1)$$

$$\rightsquigarrow \alpha = (-1)^k$$

$$\left(j_{X*}\, p_E^{*} a \cdot j_{X*}\, p_E^{*} b\right)_M = (-1)^k (a \cdot b)_X, \text{ same for } Y.$$

4) Non birationality

Standard argument: X, Y CY Picard number one on birational $\Rightarrow$ isomorphic

$\rightsquigarrow$ we just need to show $X \not\cong Y$.

Sketch of the main parts of the proof:

I. Prove that an isomorphism $f : X \to Y$ induces an automorphism φ_f of $H^0(F, \mathcal{O}(1,1))$ fixing σ

- If $X = Z(s) = Z(s')$ for $s, s' \in H^0(G(U,V), Q^\vee(2))$ then $s = \lambda s'$ w/ $\lambda \in \mathbb{C}^*$
- If $X = Z(s) \subset G(U,V)$ and $X = Z(s') \subset gG(U,V)$ then $Q_{G(U,V)}|_X \simeq Q_{gG(U,V)}|_X$ (by stability + $\exists$ morphism)
- Iso class of restr. of quotient determines $X \hookrightarrow \mathbb{P}(\Lambda^2 V)$

$\leadsto$ X contained in a unique translate of $G(U,V)$ as a zero locus of $Q^\vee(2)$.

- This implies $f : X \xrightarrow{\simeq} Y$ must be induced by an isomorphism $G(k,V) \simeq G(k+1,V)$ (call it f)

choose $V \xrightarrow{T_f} V^\vee$ $\rightsquigarrow$

$$\begin{tikzcd} G(k,V) \arrow[r, "T_f"] \arrow[rr, bend right, "f = D \circ T_f"'] & G(k,V^\vee) \arrow[r, "D"] & G(k+1,V) \end{tikzcd}$$

$$\begin{tikzcd} G(k+1,V) \arrow[r, "D^\vee"] \arrow[rr, bend right, "f^\vee"'] & G(k,V^\vee) \arrow[r] & G(k,V) \end{tikzcd}$$

These maps all extend to $\mathbb{P}(\Lambda^2 V)$, so we define:

- $\varphi_f : \mathbb{P}(\Lambda^2 V) \times \mathbb{P}(\Lambda^2 V) \longrightarrow \mathbb{P}(\Lambda^2 V) \times \mathbb{P}(\Lambda^2 V)$

  $$x, y \longmapsto (f^\vee)^{-1}(y), f(x)$$

- $\psi_f : H^0(\mathbb{P}(\Lambda^2 V) \times \mathbb{P}(\Lambda^2 V), \mathcal{O}(1,1)) \to H^0(\mathbb{P}(\Lambda^2 V) \times \mathbb{P}(\Lambda^2 V), \mathcal{O}(1,1))$

  $$\sigma \longmapsto \sigma \circ \varphi_f$$

II. $\nexists \psi_f \Leftarrow \nexists A : G(k,V) \xrightarrow{\sim} G(k+1,V)$ s.t. $\sigma A = A\sigma^T$

a $O(1,1)$-section σ is a $4c \times lo$ matrix

$\rightsquigarrow \psi_f : x^T \sigma y \longmapsto ((f^\vee)^{-1}(y))^T \sigma f(y)$

$= \ldots = y^T A \sigma^T A^{-1} x$

for A = matrix representing f.

III. Obs. that A can't exist if we prove the following:

<u>CLAIM</u> $\sigma, \sigma^T \in PGL(\Lambda^k V)$ lie in different $PGL(V) \times PGL(V)$ - orbits wrt the action:

$$PGL(V) \times PGL(V) \times PGL(\Lambda^k V) \longrightarrow PGL(\Lambda^k V)$$

$$[\sigma], [A], [B] \longmapsto [\Delta(A)^{-1} \sigma \Delta(B)]$$

for Δ = matrix of k-minors

[Manivel '10] $\rightsquigarrow$ $GL(V)$ has a dense orbit in $F(1, 2, \dots, \Lambda^k V)$